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DETERMINING THE SHAPE OF AN AXISYMMETRIC BODY IN A VISCOUS INCOMPRESSIBLE FLOW ON THE BASIS OF THE PRESSURE DISTRIBUTION ON THE BODY SURFACE

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A method is developed for determining the shape of an axisymmetric body on the basis of the pressure coefficient distribution specified along the meridional section of the body. Viscosity is taken into account within the framework of the boundary layer model. The method is based on an iterative process, which involves the solutions of the inverse problem in the plane case and of the direct problem for an axisymmetric body. A code implementing the iterative process is written, and examples of numerical results are given.

Key words: *inverse boundary-value problem of aerohydrodynamics, viscous incompressible fluid, boundary layer, axisymmetric body, iterative process, panel method.*

Introduction. Inverse boundary-value problems of aerohydrodynamics (IBPAs), which are incorporated into the general theory of inverse boundary-value problems, are used to determine the shape of an airfoil on the basis of the velocity or pressure distribution specified on the airfoil surface. In the case of spatial flows, inverse boundary-value problems can be used to design airships and to optimize the shapes of the fore and aft parts of flying vehicles.

One of the basic idealizing models of aerodynamics, which simplify the calculations of aircraft wings, is the model of plane sections, where the body cross sections are considered instead of the body as a whole. Both direct problems of this kind (see, e.g., [1–3]) and inverse problems (see, e.g., [4]) were studied in detail. Methods that allow obtaining numerical-analytical solutions for various fluid models [ideal incompressible fluid (IIF) or viscous compressible fluid] were developed. If it is impossible to use analytical methods for airfoil calculations, then numerical methods are applied. The model of plane sections, however, is inapplicable for three-dimensional bodies with complicated geometry. In this case, direct problems are solved by numerical methods where a grid covering the entire body surface is constructed. Inverse spatial problems are also solved by numerical methods, which are mainly iterative methods and, therefore, can be applied in particular cases only. For instance, with the use of the model of meridional sections, the flow around axisymmetric bodies can be considered as a two-dimensional problem: construction of bodies of revolution on the basis of a specified chord diagram of the velocity distribution [5] or on the basis of a specified pressure distribution [6], determination of the shape of turbomachinery blades located on an axisymmetric stream surface [7], and construction of an axisymmetric body on the basis of a specified velocity distribution on the body surface [8].

The present work deals with solving the problem of an axisymmetric viscous incompressible fluid (VIF) flow around an axisymmetric body, in contrast to [5, 6, 8] where the study is based on the IIF model. Viscosity is taken into account within the framework of the boundary layer (BL) model. The iterative process constructed for solving this problem includes methods of solving both inverse and direct problems.

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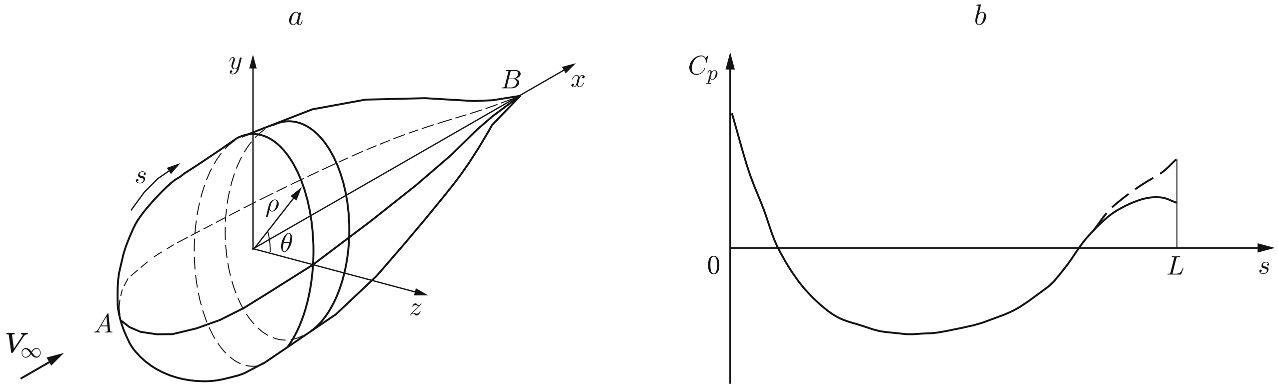

Fig. 1. Formulation of the problem: (a) sought axisymmetric body; (b) distribution of the pressure coefficient along the meridional section $C_p(s)$ (the solid and dashed curves refer to an axisymmetric and symmetric bodies, respectively).

1. Formulation of the Problem. An impermeable axisymmetric body is placed into an axisymmetric VIF flow with a specified velocity at infinity $\boldsymbol{V}_\infty$ (Fig. 1a). The body is defined by the contour of its meridional section; the axis of symmetry of the body and the free-stream velocity vector are aligned with the x axis. The distribution of the pressure coefficient $C_p = C_p(s)$ ($s \in [0, L]$) is specified on the contour (Fig. 1b). The arc coordinate s of the contour of the meridional section is counted from the point A ($s = 0$), which is the point of flow branching, to the point B on the trailing edge ($s = L$). The Reynolds number (Re) is also specified in advance. The task is to determine the shape of the meridional section of the body.

2. Numerical-Analytical Method of the Solution. An effective approach to solving the problem posed is the approach developed in [8], where an iterative process of solving the inverse problem for an axisymmetric body is constructed in accordance with the IIF model with the use of methods of solving the inverse problem for a plane contour and the direct problem for an axisymmetric body. The calculations performed revealed rapid convergence of the process (6–8 iterations on the average) with the accuracy reaching 10^{-6}. Let us construct a similar iterative process by using methods of solving the inverse plane problem and the direct axisymmetric problem, based on the VIF model.

1. Let us choose the initial approximation of the distribution of the pressure coefficient $C_{p1}^{(0)}(s)$ for solving the IBPA in the plane case: $C_{p1}^{(0)}(s) = C_p(s)$ and $k = 0$.

2. Based on the distribution $C_{p1}^{(k)}(s)$, we find the contour $L_z^{(k)}$ by solving the IBPA for a symmetric airfoil.

3. If the constructed contour $L_z^{(k)}$ is open, we close it by the method of the quasi-solution [4]. If the contour has internal intersections with itself, then the airfoil shape is corrected to obtain a univalent profile.

4. Let us define $C_{p2}^{(k)}(s)$ as the solution of the direct boundary-value problem for an axisymmetric body whose meridional section is the contour $L_z^{(k)}$ with a specified free-stream velocity V_∞.

5. Let us calculate the residue $\delta^{(k)}(s) = C_p(s) - C_{p2}^{(k)}(s)$.

6. The next approximation of the pressure coefficient distribution for the plane IBPA is determined by the formula $C_{p1}^{(k+1)}(s) = C_{p1}^{(k)}(s) + \lambda\delta^{(k)}(s)$, where $\lambda \in [0, 1]$ is the relaxation coefficient.

7. We assume that $k = k + 1$ and pass to step No. 2.

The criterion of finalizing the iterative process is the condition $\|\delta^{(k)}(s)\| < \mu$ (μ is a specified positive number). In this case, the solution obtained coincides with specified accuracy with the initial distribution of the pressure coefficient. As the inverse problem may fail to have a solution in the class of closed contours in the general case, the second condition of finalizing the process is $\|C_{p2}^{(k+1)}(s) - C_{p2}^{(k)}(s)\| < \mu$. In this case, the distribution of the pressure coefficient on the resultant airfoil can differ from the specified distribution.

It should be noted that only the condition of a closed airfoil contour was used in solving the plane IBPA by the method of the quasi-solution, and the free-stream velocity obtained in the course of the solution could be different from the initially specified velocity V_∞. The direct boundary-value problem of aerohydrodynamics was solved with the initially specified free-stream velocity. If the resultant contour has internal intersections with itself,

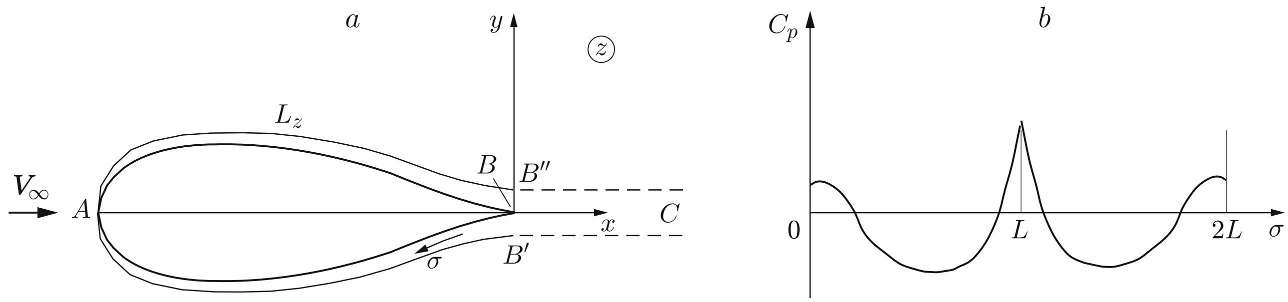

Fig. 2. Solving the IBPA in the plane case: (a) physical plane z; (b) distribution of the pressure coefficient over the airfoil contour.

the airfoil univalence can be obtained by increasing the slope of the tangent line to the curve $C_{p1}^{(k)}(s)$ in the vicinity of the trailing edge (dashed curve in Fig. 1b).

3. Solution of the Inverse Boundary-Value Problem for a Symmetric Airfoil. In solving the inverse boundary-value problem in the plane case, we have to determine the shape of the symmetric airfoil L_z (Fig. 2a) in a VIF flow without separation on the basis of the pressure coefficient distribution $C_p = C_p(\sigma)$ ($\sigma \in [0, 2L]$) specified on the airfoil surface (Fig. 2b). The arc coordinate σ of the airfoil contour is counted from the point B so that the flow region is on the left with increasing σ. The point A with the arc abscissa $\sigma = \sigma_* = L$ is the point of flow branching.

In solving the IBPA by the BL model, the problem is reduced to finding the fictitious airfoil $CB'AB''C$ on the basis of the velocity distribution $V = V(\sigma)$ specified on the segment $B'AB''$ [4]. The dependence $V(\sigma)$ is determined on the basis of the specified function $C_p(\sigma)$ in the formula following from the Bernoulli integral:

$$V(\sigma) = V_\infty \sqrt{1 - C_p(\sigma)}.$$

The problem of finding the fictitious airfoil is similar to the basic IBPA for an airfoil in an IIF flow. If the sought airfoil and the flow around it are symmetric, the method of solving the inverse problem is simplified.

As the flow contains no vortices, there exists a velocity potential $\varphi(x, y)$ and a corresponding stream function $\psi(x, y)$. Let us introduce into consideration a complex flow potential $w(z) = \varphi(x, y) + i\psi(x, y)$. The assumption of non-permeability of the fictitious airfoil contour implies that $\psi = 0$ on this contour. Then, assuming that $\varphi(\sigma_*) = 0$ on the fictitious airfoil surface, we obtain

$$\varphi(\sigma) = \int_{\sigma_*}^{\sigma} V(\sigma)\, d\sigma, \qquad \sigma \in [0, 2L]. \tag{1}$$

Let us denote $\varphi_1 = \varphi(2L)$. As we consider a symmetric flow around a symmetric body, the circulation is $\Gamma = 0$.

Let us introduce into consideration a canonical plane $\zeta = r\,\mathrm{e}^{i\gamma}$ where the region of the flow around the fictitious airfoil corresponds to the exterior of a circle of unit radius $|\zeta| \geqslant 1$ with a cut from the point $\zeta = 1$ to infinity. For mutually single-valued conformal mapping of these regions, we require the correspondence of the infinitely remote points of the planes z and ζ, and also the correspondence of the point $\zeta = 1$ to the point $z = 0$. In the course of mapping, the boundary of the unit circle $|\zeta| = 1$ transforms to the fictitious airfoil boundary, and the upper and lower edges of the cut transform to the lines $B''C$ and $B'C$, respectively. The complex potential of the symmetric flow in the canonical plane has the form

$$w(\zeta) = u_0(\zeta + 1/\zeta) + C_0,$$

where u_0 is the absolute value of velocity at infinity in the plane ζ and C_0 is a constant.

Assuming that $\zeta = \mathrm{e}^{i\gamma}$ and taking into account that the velocity potential $\tilde{\varphi}(\gamma) = \mathrm{Re}\,[w(\mathrm{e}^{i\gamma})]$ at the points $\zeta = 1$ and $\zeta = -1$ is known [$\tilde{\varphi}(0) = \varphi_1$ and $\tilde{\varphi}(\pi) = 0$], we obtain a system of equations for finding the unknown parameters. Solving this system, we find $C_0 = 2u_0 = \varphi_1/2$. Finally, we have the following equation on the boundary L_ζ:

$$\tilde{\varphi}(\gamma) = \varphi_1(\cos\gamma + 1)/2, \qquad \gamma \in [0, 2\pi]. \tag{2}$$

As the values of the velocity potentials $\varphi(\sigma)$ and $\tilde{\varphi}(\gamma)$ at the corresponding points in the planes z and ζ coincide, Eqs. (1) and (2) yield the dependence $\sigma = \sigma(\gamma)$, $\gamma \in [0, 2\pi]$.

Let us introduce into consideration an auxiliary function, which is analytical in the domain $|\zeta| \geqslant 1$:

$$\chi(\zeta) = \ln \frac{dz}{d\zeta} = \ln \left|\frac{dz}{d\zeta}\right| + i \arg \left(\frac{dz}{d\zeta}\right) = \xi + i\eta.$$

Then, we find the boundary value of its real part $\xi(\gamma) = \ln |ds/d\gamma|$ for $\zeta = \mathrm{e}^{i\gamma}$.

As the fictitious airfoil should not have inflections at the points B' and B'', i.e., the internal angle with respect to the flow region is equal to π, and the contour L_ζ in the neighborhood of the image of these points in the plane ζ has the angle $\pi/2$, then we have $z(\zeta) \sim (1 - 1/\zeta)^2$ in this neighborhood. Hence, the function $\chi(\zeta)$ has a logarithmic singularity at the point $\zeta = 1$. Eliminating this singularity, we obtain the function

$$\tilde{\chi}(\zeta) = \chi(\zeta) - \chi_0(\zeta) = \ln \frac{dz}{d\zeta} - \ln \left(1 - \frac{1}{\zeta}\right) = P + iQ.$$

Knowing the function $\sigma = \sigma(\gamma)$, we find $P(\gamma) = \mathrm{Re}\,[\tilde{\chi}(\gamma)]$ on the boundary L_ζ and, using the Schwarz formula, reconstruct the function $\tilde{\chi}(\zeta)$ with the density $P(\gamma)$:

$$\tilde{\chi}(\zeta) = -\frac{1}{2\pi} \int\limits_0^{2\pi} P(\tau)\, \frac{\mathrm{e}^{i\tau} + \zeta}{\mathrm{e}^{i\tau} - \zeta}\, d\tau.$$

The function $z(\zeta)$ is found by the formula

$$z(\zeta) = \int\limits_1^{\zeta} \left(1 - \frac{1}{\zeta}\right) \mathrm{e}^{\tilde{\chi}(\zeta)}\, d\zeta. \tag{3}$$

Substituting the limiting value $\zeta = \mathrm{e}^{i\gamma}$ into Eq. (3), we obtain the parametric equations for the sought contour of the fictitious airfoil

$$x(\gamma) + iy(\gamma) = \int\limits_0^{\gamma} 2\sin(\tau/2)\, \mathrm{e}^{P(\gamma) + i\Theta(\gamma)}\, d\tau, \qquad \gamma \in [0, 2\pi],$$

where $\Theta(\tau) = Q(\tau) + \pi + \tau/2$.

To determine the shape of the airfoil proper, we have to move normal to the fictitious airfoil boundary on the segment $B'AB''$ inward this fictitious airfoil at a distance equal to the displacement thickness $\delta_1(\sigma)$, which can be found by boundary layer calculations with an arbitrary available method (see, e.g., [2, 9–11]).

The airfoil obtained as a result of this solution can be physically non-feasible, i.e., fail to satisfy the conditions of solvability (existence and uniqueness) and univalence.

The existence of a physically feasible airfoil implies that its contour is closed. For the BL model, the closed contour of the airfoil means that the fictitious airfoil is non-closed by $\delta_1^0 = 2\delta_1(0) = 2\delta_1(2L)$. The airfoil contour is closed if the following equality is valid:

$$\int\limits_0^{2\pi} P(\tau) \cos\tau\, d\tau = -\pi + \frac{V_\infty \delta_1}{2u_0}.$$

To satisfy the solvability conditions, we use the method of the quasi-solution [4]. In addition, the resultant contour should not have intersections with itself.

4. Solution of the Direct Boundary-Value Problem for an Axisymmetric Body. In solving the direct boundary-value problem, we have to determine the distribution of the pressure coefficient along the contour of the meridional section $C_p(s)$, $s \in [0, L]$ for an axisymmetric body in a VIF flow with a specified velocity at infinity V_∞. The axis of symmetry of the body and the free-stream velocity vector are aligned with the x axis (see Fig. 1a).

The direct problem in the axisymmetric case is solved by the panel method (see, e.g., [1, 12]). Rings located along the body of revolution are used as panels. Let us pass to the cylindrical coordinates x, ρ, θ related to the Cartesian coordinates x, y, z as $y = \rho \sin\theta$ and $z = \rho\cos\theta$ and consider the meridional section at $\theta = 0$. Then, the

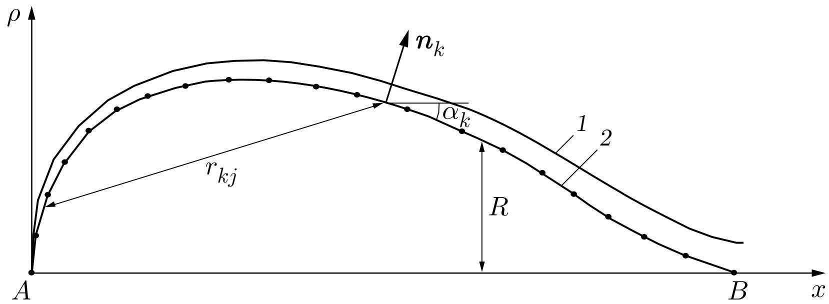

Fig. 3. Shape of the meridional section of the axisymmetric body obtained by solving the direct problem: fictitious airfoil contour (1) and axisymmetric body contour (2).

panels in the meridional section of the body are consecutively connected segments $\{x_j, \rho_j\}$, $j = \overline{1, N+1}$, where N is the number of panels. The middles of the panels with the coordinates $\{x_{jc} = (x_j + x_{j+1})/2,\ \rho_{jc} = (\rho_j + \rho_{j+1})/2\}$ ($j = \overline{1, N}$) are chosen as the control points.

It is assumed that hydrodynamic singularities (sources and sinks) are distributed over the body surface. Let us denote the intensity of the distributed sources of the jth panel by q_j ($q_j > 0$ for the source and $q_j < 0$ for the sink); the intensity on each panel is assumed constant along the panel. The values of q_j are found in the course of solving the problem. Let us introduce the following notation: α_j for the angle between the jth panel and the positive direction of the x axis ($a_j = \tan\alpha_j$) and r_{kj} for the distance between the current point of integration of the jth panel with singularities and the control point of the k-th panel (Fig. 3).

The velocity potential $\Phi(x, \rho)$ is composed of the free-stream potential $\Phi_\infty = V_\infty x$ and the sum of the potentials $\Phi_j(x, \rho)$ induced by hydrodynamic singularities on the panels. The potential at the kth point has the form

$$\Phi(x_k, \rho_k) = V_\infty x_k + \sum_{j=1}^{N} \Phi_j(x_k, \rho_k) = V_\infty x_k - \sum_{j=1}^{N} \frac{q_j}{4\pi} \int \frac{dS}{r_{kj}}. \tag{4}$$

Equation (4) is written in the general form and can be used with an arbitrary choice of the panels and the method of distributing the singularities. The normal and tangential components of velocity are expressed via the derivatives of the potential:

$$V_n(x_k, \rho_k) = \frac{\partial \Phi(x_k, \rho_k)}{\partial n} = \frac{\partial \Phi(x_k, \rho_k)}{\partial x}\, n_{xk} + \frac{\partial \Phi(x_k, \rho_k)}{\partial \rho}\, n_{\rho k}; \tag{5}$$

$$V_s(x_k, \rho_k) = \frac{\partial \Phi(x_k, \rho_k)}{\partial n} = \frac{\partial \Phi(x_k, \rho_k)}{\partial x}\, s_{xk} + \frac{\partial \Phi(x_k, \rho_k)}{\partial \rho}\, s_{\rho k}. \tag{6}$$

Here $n_{xk} = -s_{\rho k} = -\sin\alpha_k$ and $n_{\rho k} = s_{xk} = \cos\alpha_k$.

In the present work, we consider the case where the hydrodynamic singularities are located directly on the panels. The expression for the velocity potential at the k-th control point is written as

$$\Phi(x_k, \rho_k) = V_\infty x_k - \sum_{j=1}^{N} \frac{q_j\sqrt{1 + a_j^2}}{4\pi} \int_0^{2\pi} \int_{x_{j1}}^{x_{j2}} \frac{\rho_j\, d\theta\, dx_j}{\sqrt{(x_k - x_j)^2 + \rho_k^2 + \rho_j^2}}.$$

Then, the expression for the normal component of velocity acquires the form

$$V_n = V_\infty n_x - \frac{1}{4\pi} \sum_{j=1}^{N} q_j \int_{x_{j1}}^{x_{j2}} \Bigg(\frac{2\rho_j\sqrt{1 + a_j^2}\, (E(m_{kj}) - K(m_{kj}))n_\rho}{\rho_k\sqrt{(\rho_k - \rho_j)^2 + (x_k - x_j)^2}}$$

$$- \frac{4\rho_j\sqrt{1 + a_j^2}\, ((\rho_k + \rho_j)n_\rho + (x_k - x_j)n_x)E(m_{kj})}{((\rho_k + \rho_j)^2 + (x_k - x_j)^2)\sqrt{(\rho_k - \rho_j)^2 + (x_k - x_j)^2}} \Bigg) dx_j, \tag{7}$$

where $K(m_{kj})$ is the complete elliptical integral of the first kind, $E(m_{kj})$ is the complete elliptical integral of the second kind, and $m_{kj} = -4\rho_k\rho_j/\sqrt{(\rho_k - \rho_j)^2 + (x_k - x_j)^2}$. Thus, the integral in Eq. (7) has a singularity if the point k coincides with the point j. To avoid the singularity of the integral at $k = j$, the control points should be shifted into the flow by a distance ε from the airfoil contour. The coordinates of these points are $x'_k = x_k + \varepsilon n_{xk}$ and $\rho'_k = \rho_k + \varepsilon n_{\rho k}$ [3]. The integrals proper are found numerically with the use of quadrature formulas. The experience of calculating an IIF flow around a sphere shows that the coincidence with the analytical solution ($\|\delta\| < 10^{-8}$) is reached at distances from the contour equal to 0.01–1.00 of the panel length. In all calculations, the value of ε was taken to be 0.1 of the panel length.

If the panel method is used in the IIF case, the values of q_j are calculated from the condition of flow non-penetration through the body surface:

$$V_n(x_k, \rho_k) = 0.$$

In the case of the viscous fluid flow, the condition of non-penetration through the surface of the fictitious airfoil is used in accordance with the BL model. In this case, the normal component of velocity on the body surface is not equal to zero:

$$V_n(x_k, \rho_k) = V_n \neq 0. \tag{8}$$

The value of V_n can be found from the continuity equation for the boundary layer on an axisymmetric body [12, 13]

$$V_n = \frac{d\,(U_e\delta_1)}{ds} + \frac{U_e\delta_1}{R}\,\frac{dR}{ds}, \tag{9}$$

where $U_e(s)$ is the velocity distribution on the surface of the fictitious airfoil in the IIF flow, $\delta_1(s)$ is the distribution of the BL displacement thickness, and $R(s)$ is the radius of the meridional section.

Expression (8) is a system of linear algebraic equations with respect to unknown q_j, which is used to find the velocity distribution on the body surface. The sought velocity distribution on the surface of the fictitious airfoil $U_e(s)$ and the distribution $\delta_1(s)$ depending on this velocity distribution are included into the right side of expression (8); hence, the following iterative process is constructed for determining $U_e(s)$.

1. The initial approximation $U_e^{(n)}(s)$ $(n = 0)$ is taken from the solution of the problem within the framework of the IIF model.

2. System (8) is solved with allowance for Eq. (9).

3. After the potential $\Phi^{(n)}(x_k, \rho_k)$ is determined, the velocity components $V_s^{(n)}(x_k, \rho_k)$ along the body contour are found by Eq. (6).

4. The next approximation $U_e^{(n+1)} = V_s^{(n)}$ is specified.

5. It is assumed that $n = n + 1$, and the procedure is returned to step No. 2.

The process is continued until the condition $|U_e^{(n+1)}(s) - U_e^{(n)}(s)| < \mu$ is satisfied. Then, $U_e^{(n+1)}(s)$ is the sought distribution of velocity on the surface of the fictitious airfoil. The distribution of the pressure coefficient over the body surface can be found by the formula following from the Bernoulli integral:

$$C_p(\sigma) = 1 - (V(\sigma)/V_\infty)^2.$$

In flow calculations for an axisymmetric BL, the laws of conservation of mass and momentum are written as

$$\begin{gathered} \frac{\partial\,(r^k u)}{\partial s} + \frac{\partial\,(r^k v)}{\partial n} = 0, \\ u\,\frac{\partial u}{\partial s} + v\,\frac{\partial u}{\partial n} = U\,\frac{\partial U}{\partial s} + \frac{\partial}{\partial n}\Big(r^k \nu\,\frac{\partial u}{\partial n}\Big), \end{gathered} \tag{10}$$

where u and v are the components of the velocity vector, U is the velocity on the boundary layer edge, ν is the kinematic viscosity, and r is the distance from the axis of symmetry to the point in the boundary layer. System (10) describes a plane BL (at $k = 0$) or an axisymmetric BL (at $k = 1$). The following boundary conditions are set:

$$u(s, 0) = v(s, 0) = 0, \qquad \lim_{n\to\infty} u(s, n) = U(s).$$

As the boundary layer in system (10) is thin, then the values of $r(s, n)$ can be replaced by $R(s)$, which substantially simplifies further calculations. This replacement can restrict the range of applicability of the BL

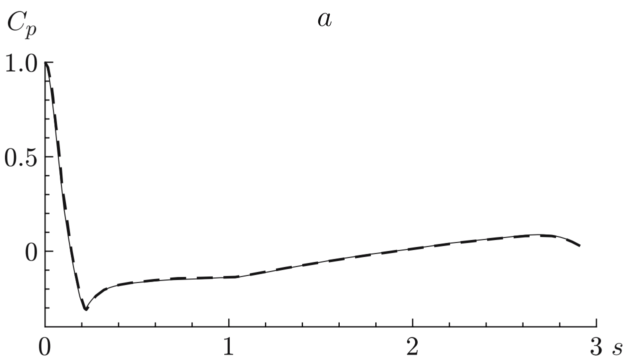

Fig. 4. Distribution of the pressure coefficient along the meridional section of the body obtained in test calculations: (a) body with the thickness smaller than 15%; (b) body with the thickness greater than 15%; the solid curves are the initial distributions; the dashed curves are the distributions obtained by solving the IBPA, and the points are the solutions obtained by the Fluent CFD code.

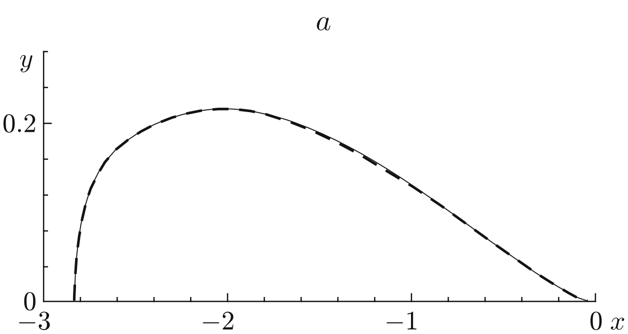

Fig. 5. Shape of the meridional section of the body obtained in test calculations: (a) body with the thickness smaller than 15%; (b) body with the thickness greater than 15%; the solid curves show the initial shape; the dashed curves show the shape obtained by solving the IBPA.

equations (e.g., in the trailing edge, where the boundary layer thickness is sufficiently large, and the radius of the meridional section tends to zero). Nevertheless, the present numerical calculations, as well as the calculations and comparisons with experimental data in [12, 13], showed that this assumption can also be used for extremely small values of $R(s)$, and the values at the last points can be obtained by extrapolation.

Equations of an axisymmetric BL can be reduced to equations of a plane BL by the Mangler–Stepanov transformation (see, e.g., [9, 10]):

$$\bar{s} = \int\limits_0^s R^2(\xi)\,d\xi, \quad \bar{n} = R(s)n, \quad \bar{u} = u, \quad \bar{v} = \frac{v}{R} + \frac{R'nu}{R^2}, \quad \bar{U} = U.$$

Further BL calculations are performed for transformed coordinates by the Kochin–Loitsyanskii method (see, e.g., [2, 9, 10]) or by the Eppler method (see, e.g., [11, 12]).

The drag coefficient is calculated by an analog of the Squire–Young formula for an axisymmetric body [12, 13]

$$C_x = \frac{4\pi R(s)\delta_2(s)}{V^{2/3}}\Big(\frac{U_e(s)}{U_\infty}\Big)^{(H_{12}+5)/2}, \tag{11}$$

where $\delta_2(s)$ is the momentum loss thickness, V is the body volume, U_∞ is the velocity at infinity, and $H_{12} = \delta_1/\delta_2$ is the shape parameter. In contrast to the plane case, the drag coefficient of an axisymmetric body is assumed to be $\max C_x(s)$, because $R(s) \to 0$ in the trailing edge.

5. Test Calculations. To test the method described above, we performed numerical calculations whose results are plotted in Figs. 4 and 5. The initial distributions of the pressure coefficient were taken to be those obtained by solving the direct problem for the bodies considered (solid curves in Fig. 4). In all calculations, we

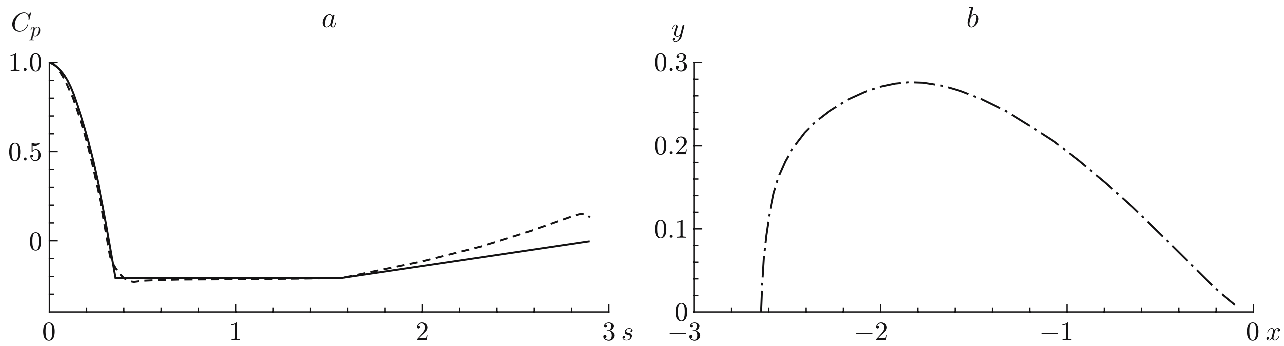

Fig. 6. Results of solving the problem of determining the shape of an axisymmetric body: (a) distribution of the pressure coefficient along the meridional section of the body (the solid curve is the specified distribution and the dashed curve is the distribution obtained by solving the IBPA); (b) shape of the meridional section of the body.

used $\mathrm{Re} = 10^6$. All the bodies considered have a sharp trailing edge with the outer angle equal to 2π, and the flow is not separated there. For bodies with the thickness smaller than 15% of the chord length (Fig. 5a), the iterative process rapidly converges (4–6 iterations on the average) with accuracy of $\|\delta\| \approx 10^{-6}$. The shape of the meridional section obtained in calculations and the corresponding distribution of the pressure coefficient are shown by the dashed curves in Figs. 4 and 5.

In addition to slender bodies, we considered bodies of large thickness (solid curve in Fig. 5b). For the body in a flow without separation, the distribution of the pressure coefficient predicted by the solution of the direct boundary-value problem of aerohydroelasticity is shown by the solid curve in Fig. 4b. Nevertheless, the BL calculations predict that the flow becomes separated already at the third iterative step in solving the IBPA. The separation occurs in the aft part of the body, owing to a negative velocity gradient. It is impossible, therefore, to obtain the meridional section of the initial body, because the IBPA with allowance for viscosity has a solution only under the assumption of a non-separated flow. One possible method to overcome this difficulty is solving the plane IBPA with the use of the IIF model. In other words, the result of calculations based on a specified distribution of the pressure coefficient is a symmetric airfoil in an IIF flow, rather than a fictitious airfoil. The resultant meridional section and the corresponding distribution of the pressure coefficient are plotted by the dashed curves in Figs. 4b and 5b. The iterative process converges after a greater number of iterations (20 iterations). The reason is a more pronounced effect of viscosity in the aft part of the body. The solution with allowance for the BL presence makes it possible to determine the angle and the thickness of the trailing edge more exactly. The use of the IIF model (for the IBPA solution to satisfy the closure condition) reduces the accuracy of this solution, while the allowance for viscosity in solving the inverse plane problem ensures faster convergence of the iterative process with higher accuracy.

To estimate the reliability of the results obtained, we also performed calculations by the Fluent CFD code. For this purpose, regular rectangular grids were generated by the Gambit grid generator. The calculations were performed with the Spalart–Allmaras model of turbulence. The distribution of the pressure coefficient obtained for the example shown in Fig. 5b is plotted by the points in Fig. 4b. An analysis of the results shows that the distributions of the pressure coefficient are in good agreement ($\|\delta\| \approx 10^{-8}$).

The drag coefficient is $C_x = 0.026$ for the initial body and $C_x = 0.025$ for the designed body. The close values of the drag coefficients are obtained in the calculations because the area $S = V^{2/3}$ is used as the characteristic scale in Eq. (11). In other words, if the geometry of the bodies is similar, the changes in the aerodynamic characteristics R, δ_2, and U_e in Eq. (11) are compensated by the changes in the area of the meridional section.

6. Example of the IBPA Solution for an Axisymmetric Body. As the body geometry is not known in advance in solving inverse problems, it is necessary to determine the shape of the meridional section for a relatively arbitrary distribution of the pressure coefficient, i.e., for a distribution that does not correspond to the specified meridional section. Moreover, the pressure distribution in calculations of a viscous fluid flow should correspond to the flow regime with no separation.

The solid curve in Fig. 6 shows the initial distribution of the pressure coefficient; the contour of the meridional section and the corresponding distribution of the pressure coefficient obtained by solving the IBPA are plotted by the dot-and-dashed and dashed curves, respectively. The drag coefficient is $C_x = 0.026$. The pressure coefficient in the aft part differs from the initial value because it was necessary to satisfy the solvability conditions. As inverse problems in the general case may fail to have any solutions, we can argue that this means that there is no physically feasible body for a specified distribution of the pressure coefficient and a specified free-stream velocity.

Conclusions. Thus, the solution of the problem of constructing an axisymmetric body in an IIF flow is extended to the case with allowance for viscosity by the BL model, which makes it possible to determine the value of C_x. An analysis of results of numerical calculations performed confirms the effectiveness of the iterative method of solving the IBPA for axisymmetric bodies in a VIF flow. As the method of solving inverse problems and the numerical method of solving direct problems are used in the process, it is difficult to estimate the convergence of the method analytically. The numerical calculations performed, however, allow us to conclude that the proposed iterative method converges rather rapidly (on the average, within 20 iterations or less). The use of the panel method for modeling the viscous flow in solving the direct problem showed that this method is highly efficient, which ensures a high speed of codes implementing the method developed.

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